Acta Sci. Math., 43 (1981), 55—64

Imbedding theorems and strong approximation

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1. Several recent papers, e.g. [2], [3], [4], [8], investigate problems of the following type: Under what conditions can a certain class of functions be imbedded in another class, where at least one of these classes is determined by certain properties of the strong approximation of Fourier series.

Our aim is also to prove two theorems of this type.

Before formulating them we give the definitions and notations used in the paper and draw some background.

Let $f(x)$ be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series. Denote by $s_n = s_n(x) = s_n(f, x)$ the n-th partial sum of (1) and by $f^{(r)}$ the r-th derivative of f.

Let $\omega(\delta)$ be a non-decreasing continuous function on the interval $[0, 2\pi]$ having the properties: $\omega(0)=0$, $\omega(\delta_1+\delta_2) \leqq \omega(\delta_1)+\omega(\delta_2)$ for any $0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1+\delta_2 \leqq 2\pi$. Such a function will be called a *modulus of continuity*.

Let $E_n(f)$ denote the best approximation of f by trigonometric polynomials of order at most n.

We define the following class of functions:

$$W^r H^\omega := \{f : \omega(f^{(r)}; \delta)\} = O(\omega(\delta)),$$

where $\omega(f; \delta)$ is the modulus of continuity of f. In the case $\omega(\delta)=\delta^\alpha$ we write $W^r H^\alpha$ instead of $W^r H^{\delta^\alpha}$; and if $r=0$ H^ω stands for $W^0 H^\omega$, and in many cases Lip 1 will denote the class H^1.

Generalizing a theorem of SZABADOS [7] we proved in ([6]) the following result:

Received December 17, 1979.

If $0<\alpha\leqq 1$, $p>0$ *and* r *is a nonnegative integer, then*

$$\left\|\sum_{n=1}^{\infty} n^{(r+\alpha)p-1}|s_n-f|^p\right\|<\infty \tag{2}$$

implies that

$$\omega(f^{(r)};\delta)=\begin{cases} O\left(\delta\log\frac{1}{\delta}\right) & \text{if } \alpha=1,\\ O(\delta^{\alpha}) & \text{if } 0<\alpha<1,\end{cases}$$

where $\|\cdot\|$ *denotes the usual maximum norm. These estimations are best possible.*

On account of this result it is clear that condition (2) with $\alpha=1$ does not imply that $f^{(r)}\in \mathrm{Lip}\,1$. But the following condition

$$\sum_{m=0}^{\infty}\left\|\left\{\sum_{n=2^m+1}^{2^{m+1}} n^{(r+1)p-1}|s_n-f|^p\right\}^{1/p}\right\|<\infty, \tag{3}$$

which claims just a little bit more than (2) with $\alpha=1$ does, is already sufficient for $f^{(r)}$ to belong to the class Lip 1 (see [5], Theorem 5). Thus it is natural to ask whether condition (3) is also necessary for $f^{(r)}\in \mathrm{Lip}\,1$. We shall prove that the answer to this question is negative, but condition (3) cannot be weakened in general. Indeed, the following more general theorem also holds.

Theorem 1. *Let* $\varepsilon=\{\varepsilon_n\}$ *be a given monotone sequence. Then for any positive* p *the condition*

$$\varepsilon_n\geqq c>0 \quad (n=1,2,\ldots) \tag{4}$$

is necessary and sufficient that

$$S_p(\varepsilon,r):=\left\{f\colon \sum_{m=0}^{\infty}\varepsilon_m\left\|\left\{\sum_{n=2^m+1}^{2^{m+1}} n^{(r+1)p-1}|s_n-f|^p\right\}^{1/p}\right\|<\infty\right\}\subset W^rH^1. \tag{5}$$

Furthermore, for any sequence ε *satisfying condition* (4), *there exists a function* f_0 *such that* $f_0\in W^rH^1$ *but* $f_0\notin S_p(\varepsilon,r)$. *Thus the imbedding* (5) *is proper.*

As a special case of Lemma 6 of [6], it is also proved that (3) is equivalent to

$$\sum_{n=1}^{\infty} n^r E_n(f)<\infty.$$

Moreover, in [6] we verified that for any $p>0$ and for any positive monotone sequence $\mu=\{\mu_n\}$ with the property $0<k\leqq\mu_{2n}/\mu_n\leqq K<\infty$ the conditions

$$\sum_{m=0}^{\infty}\left\|\sum_{n=2^m+1}^{2^{m+1}}\mu_n|s_n-f|^p\right\|<\infty$$

and

(6) $$\sum_{n=1}^{\infty} \mu_n E_n^p(f) < \infty$$

are equivalent.

Thus it is obvious that (6) implies

(7) $$\left\| \sum_{n=1}^{\infty} \mu_n |s_n - f|^p \right\| < \infty;$$

in other words, (6) is a sufficient condition for (7), but presumably not a necessary one. We shall prove that this is the case, indeed, but we shall also verify that (6) cannot be weakened generally.

We define two further classes of functions:

$$S_p(\mu) := \left\{ f: \left\| \sum_{n=1}^{\infty} \mu_n |s_n - f|^p \right\| < \infty \right\} \quad \text{and} \quad E(\varepsilon) := \{f: E_n(f) = O(\varepsilon_n)\},$$

where $\mu = \{\mu_n\}$ and $\varepsilon = \{\varepsilon_n\}$ are given positive monotone sequences and $p > 0$.

Using these notations we can formulate our statement as follows:

Theorem 2. *Let $p > 0$ and let $\varepsilon = \{\varepsilon_n\}$ and $\mu = \{\mu_n\}$ be given positive monotone sequences such that $0 < k \leqq \mu_{2n}/\mu_n \leqq K < \infty$. In order that*

(8) $$E(\varepsilon) \subset S_p(\mu)$$

it is necessary and sufficient that

(9) $$\sum_{n=1}^{\infty} \mu_n \varepsilon_n^p < \infty.$$

If $\mu_n = n^\gamma$, $\gamma > -1$, then inclusion (8) *is proper for any $\varepsilon = \{\varepsilon_n\}$ satisfying* (9), *that is, there exists a function F such that $F \in S_p(\mu)$ but $F \notin E(\varepsilon)$.*

From Theorem 2, using a result of Krotov and Leindler [3] (see our Lemma 1), we immediately obtain

Corollary. *If there exists a positive monotone sequence $\mu = \{\mu_n\}$ such that $0 < k \leqq \mu_{2n}/\mu_n \leqq K < \infty$,*

(10) $$\sum_{n=1}^{\infty} \mu_n \varepsilon_n^p < \infty \quad \textit{and} \quad \sum_{n=1}^{m} (n\mu_n)^{-1/p} = O\left(m\omega\left(\frac{1}{m}\right)\right),$$

then $E(\varepsilon) \subset H^\omega$.

2. We require some lemmas to prove our theorems.

Lemma 1 ([3, Theorem]). *If* $0<p<\infty$ *and* $\lambda=\{\lambda_n\}$ *is a monotone sequence such that* $\{n^\theta\lambda_n\}$ *with a certain* $0<\theta<1$ *increases, then condition*

$$\sum_{\nu=1}^{n}(\nu\lambda_\nu)^{-1/p}=O\left(n\omega\left(\frac{1}{n}\right)\right) \tag{2.1}$$

is necessary and sufficient for

$$S_p(\lambda)\subset H^\omega. \tag{2.2}$$

(We mention that the assumption $n^\theta\lambda_n\uparrow$ is not needed to the proof of the implication (2.1)⇒(2.2).)

Lemma 2 ([1, see the proof of Theorem 1]). *The function*

$$g(x)=\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\sum_{l=2^{k-1}+1}^{2^k}\left(\frac{\sin(5\cdot 2^k-l)x}{l}-\frac{\sin(5\cdot 2^k+l)x}{l}\right)$$

belongs to the class Lip 1.

Lemma 3 ([6, Theorem 3 with $r=0$]). *Let* $p>0$. *Suppose that* $\{\lambda_k\}$ *is a monotonesequence satisfying the following conditions. Setting* $\Lambda_n=\sum_{k=1}^{n}\lambda_k$, $\{n^{-1}\Lambda_n\}$ *is monotone,* $\{n^{\eta-1}\Lambda_n\}$ *is non-decreasing for a certain* $\eta<1$, *and* $n\lambda_n\leqq K\Lambda_n$. *Then the function*

$$F(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n\Lambda_n^{1/p}}$$

has the following properties:

$$\left\|\sum_{n=1}^{\infty}\lambda_n|s_n(F)-F|^p\right\|<\infty \quad \text{and} \quad \omega\left(F;\frac{1}{n}\right)\geqq C\frac{1}{n}\sum_{\nu=1}^{n}\Lambda_\nu^{-1/p},$$

where C is a positive constant.

Lemma 4. *If* $a_n\geqq 0$ *and* $\sum_{n=1}^{\infty}a_n=\infty$ *then for any sequence* $\{\varepsilon_n\}$ *tending to zero there exists a monotone sequence* $\{b_n\}$ *such that* $b_n\to 0$,

$$\sum_{n=1}^{\infty}a_nb_n=\infty \tag{2.3}$$

and

$$\sum_{n=1}^{\infty}a_nb_n\varepsilon_n<\infty. \tag{2.4}$$

Proof. Since $\sum a_n=\infty$ we can define a sequence $\{\nu_m\}$ such that $\nu_1=2$, $\nu_2=4$ and if $m\geqq 3$ then

$$\sum_{n=\nu_{m-1}+1}^{\nu_m}a_n>m+\sum_{n=\nu_{m-2}+1}^{\nu_{m-1}}a_n.$$

and for any $k \geqq \nu_m$

$$\varepsilon_k < \frac{1}{m^2}.$$

From this sequence $\{\nu_m\}$ we deduce the required sequence $\{b_n\}$ as follows: Let $b_1 = b_2 = 1$ and if $\nu_{m-1} < n \leqq \nu_m$ $(m \geqq 2)$ then let $b_n = \left(\sum_{i=\nu_{m-1}+1}^{\nu_m} a_i\right)^{-1}$.

Hence an elementary calculation shows that (2.3) and (2.4) hold, and this ends the proof.

3. Proof of Theorem 1. First we prove the implication (4)$\Rightarrow$(5). If $f \in S_p(\varepsilon, r)$ and (4) holds then condition (3) is also fulfilled whence $f^{(r)} \in \mathrm{Lip}\, 1$ follows (see Theorem 5 of [5]), i.e. imbedding (5) holds.

To prove the necessity of (4) we give functions showing that (5) does not hold if $\varepsilon_n \to 0$.

First we define a new sequence $\{\varepsilon_n^*\}$ as follows: Let $\varepsilon_n^* = \varepsilon_m$ if $2^m < n \leqq 2^{m+1}$ $(m \geqq 0)$ and $\varepsilon_1^* = 1$. By Lemma 4 for the sequence $\{\varepsilon_n^*\}$ there exists a monotone sequence $\{\lambda_n\}$ such that $\lambda_n \to \infty$ and

$$\sum_{n=1}^{\infty} \frac{1}{n\lambda_n} = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\varepsilon_n^*}{n\lambda_n} < \infty. \tag{3.1}$$

Using this sequence we define the following function

$$F(x) := \sum_{n=1}^{\infty} \lambda_n^{-1} n^{-r-2} \operatorname{cvs} nx$$

where $\operatorname{cvs} x$ means $\cos x$ or $\sin x$ according as r is an odd or even integer, resp. It is clear that

$$F^{(r)}(x) = \pm \sum_{n=1}^{\infty} \lambda_n^{-1} n^{-2} \sin nx,$$

and by (3.1) $F^{(r)}$ does not belong to the class Lip 1.

On the other hand

$$|s_n(F) - F| \leqq \sum_{k=n}^{\infty} \lambda_k^{-1} k^{-r-2} \leqq K\lambda_n^{-1} n^{-r-1},$$

whence we get that

$$\left\| \left\{ \sum_{n=2^m+1}^{2^{m+1}} n^{(r+1)p-1} |s_n(F) - F|^p \right\}^{1/p} \right\| \leqq K_0 \left\{ \sum_{n=2^m+1}^{2^{m+1}} \lambda_n^{-p} n^{-1} \right\}^{1/p} \leqq \frac{K_1}{\lambda_{2^m}}.$$

Hence, on account of (3.1), we already obtain immediately that $F \in S_p(\varepsilon, r)$. As we have seen $F \notin W^r H^1$, thus (5) cannot be valid, and this proves the necessity of (4).

In order to prove that imbedding (5) is proper we consider the following functions:

If r is even then let

$$f_0(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^{r+2}},$$

and if r is odd then let

$$f_1(x) = \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \sum_{l=2^{m-1}+1}^{2^m} \left(\frac{\cos(5\cdot 2^m - l)x}{(5\cdot 2^m - l)^r l} - \frac{\cos(5\cdot 2^m + l)x}{(5\cdot 2^m + l)^r l} \right).$$

It is well known that $f_0^{(r)} \in \operatorname{Lip} 1$, and on the other hand, by Lemma 2, $f_1^{(r)} \in \operatorname{Lip} 1$ is also proved.

Thus it remains to prove that f_0 and f_1 do not belong to the class $S_p(\varepsilon, r)$ for any ε satisfying (4).

A standard calculation shows that

$$|f_0(0) - s_n(0)| \geqq cn^{-r-1}, \quad (c > 0),$$

whence

$$\left\| \left\{ \sum_{n=2^m+1}^{2^{m+1}} n^{(r+1)p-1} |s_n - f_0|^p \right\}^{1/p} \right\| \geqq \frac{c}{2}$$

follows, consequently $f_0 \notin S_p(\varepsilon, r)$.

The proof of the statement $f_1 \notin S_p(\varepsilon, r)$ needs a longer calculation. First we give a lower estimation for the difference $|f_1(0) - s_n(f_1, 0)|$ if n satisfies the inequalities $22\cdot 2^{m-2} < n \leqq 23 \cdot 2^{m-2}$ $(m \geqq 4)$. Such an n can be written in the form:

$$n = 5\cdot 2^m + l \quad \text{with} \quad 2^{m-1} < l \leqq 3\cdot 2^{m-2}.$$

Therefore, by the definition of f_1 it is easy to see that

$$|s_n(f_1; 0) - f_1(0)| \geqq \left| \frac{1}{2^m} \sum_{l=n-5\cdot 2^m+1}^{2^m} \frac{-(-1)^m}{(5\cdot 2^m + l)^r l} + \sum_{\mu=m+1}^{\infty} \frac{(-1)^\mu}{2^\mu} R_\mu \right|, \tag{3.2}$$

where

$$R_\mu = \sum_{l=2^{\mu-1}+1}^{2^\mu} \left(\frac{1}{(5\cdot 2^\mu - l)^r l} - \frac{1}{(5\cdot 2^\mu + l)^r l} \right).$$

If we show that $R_\mu \geqq R_{\mu+1} (\geqq 0)$ then by (3.2) we obtain that $|s_n(0) - f_1(0)|$ is not less than the absolute value of the first sum in (3.2), namely the sums in (3.2) have the same sign. Since $r \geqq 1$

$$\frac{1}{(5\cdot 2^{\mu+1} - 2i)^r 2i} - \frac{1}{(5\cdot 2^{\mu+1} + 2i)^r 2i} + \frac{1}{(5\cdot 2^{\mu+1} - 2i + 1)^r (2i-1)} -$$

$$- \frac{1}{(5\cdot 2^{\mu+1} + 2i - 1)^r (2i-1)} \leqq \left(\frac{1}{2i} + \frac{1}{2i-1} \right) \left(\frac{1}{(5\cdot 2^{\mu+1} - 2i)^r} - \frac{1}{(5\cdot 2^{\mu+1} + 2i)^r} \right) \leqq$$

$$\leqq \frac{2}{2^r(2i-1)} \left(\frac{1}{(5\cdot 2^\mu - i)^r} - \frac{1}{(5\cdot 2^\mu + i)^r} \right) \leqq \frac{1}{i} \left(\frac{1}{(5\cdot 2^\mu - i)^r} - \frac{1}{(5\cdot 2^\mu + i)^r} \right),$$

whence

$$R_{\mu+1} = \sum_{l=2^\mu+1}^{2^{\mu+1}} \frac{1}{l}\left(\frac{1}{(5\cdot 2^{\mu+1}-l)^r} - \frac{1}{(5\cdot 2^{\mu+1}-l)^r}\right) \leqq$$

$$\leqq \sum_{l=2^{\mu-1}+1}^{2^\mu} \frac{1}{l}\left(\frac{1}{(5\cdot 2^{\mu}-l)^r} - \frac{1}{(5\cdot 2^{\mu}+l)^r}\right) = R_\mu$$

follows obviously.

Continuing the estimate of (3.2) we have

$$|s_n(f_1, 0)-f_1(0)| \geqq \frac{1}{2^m} \sum_{l=n-5\cdot 2^m+1}^{2^m} l^{-1}(5\cdot 2^m+l)^{-r} \geqq$$

$$\geqq 6^{-r}2^{-m(r+1)} \sum_{l=n-5\cdot 2^m+1}^{2^m} l^{-1}.$$

Using this we obtain that $(m \geqq 4)$

$$\sum_{n=2^{m+2}+1}^{2^{m+3}} n^{(r+1)p-1}|s_n(0)-f_1(0)|^p \geqq \sum_{n=22\cdot 2^{m-2}+1}^{23\cdot 2^{m-2}} n^{(r+1)p-1}|s_n(0)-f_1(0)|^p \geqq$$

$$\geqq \sum_{n=22\cdot 2^{m-2}+1}^{23\cdot 2^{m-2}} n^{(r+1)p-1}6^{-rp}2^{-m(r+1)p}\left(\sum_{l=3\cdot 2^{m-2}+1}^{2^m} l^{-1}\right)^p \geqq C_{r,p} > 0,$$

where $C_{r,p}$ is independent of n.

Hence, as before, the statement $f_1 \notin S_p(\varepsilon, r)$ follows clearly, and this completes the proof of Theorem 1.

Proof of Theorem 2. *Sufficiency of condition* (9). It is clear that if $f\in E(\varepsilon)$ then (9) implies

$$\sum_{n=1}^{\infty} \mu_n E_n^p(f) < \infty,$$

and this is equivalent to

$$\sum_{m=0}^{\infty} \left\| \sum_{n=2^m+1}^{2^{m+1}} \mu_n |s_n - f|^p \right\| < \infty$$

(see Theorem 4 of [6], where the restriction on the rate of μ_{2n}/μ_n is required), whence $f\in S_p(\mu)$ follows obviously. Thus (8) holds if (9) is fulfilled.

Necessity of condition (9) will be proved indirectly. If we assume that (8) holds and

(3.3) $$\sum_{n=1}^{\infty} \mu_n \varepsilon_n^p = \infty,$$

then the function

$$f_0(x) = \sum_{n=2}^{\infty} (\varepsilon_n - \varepsilon_{n+1}) \cos(n+1)x$$

leads to a contradiction. Namely,

$$E_n(f_0) \leqq \|s_n(f_0, x) - f_0(x)\| \leqq \varepsilon_n,$$

i.e. $f_0 \in E(\varepsilon)$, but

$$|s_n(f_0, 0) - f_0(0)| = \sum_{k=n}^{\infty} (\varepsilon_k - \varepsilon_{k+1}) = \varepsilon_n,$$

whence

$$\left\| \sum_{n=2}^{N} \mu_n |s_n(f_0) - f_0|^p \right\| \geqq \sum_{n=2}^{N} \mu_n \varepsilon_n^p;$$

by (3.3) this shows that f_0 does not belong to the class $S_p(\mu)$, and this contradicts (8).

Hereby the necessity of (9) is proved.

In order to prove that inclusion (8) is strict let us consider the function

$$F(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{1+(1/p)+(\gamma/p)}}.$$

An elementary calculation shows that if $\lambda_n = n^\gamma$ $(\gamma > -1)$ then all conditions of Lemma 3 are satisfied and thus

$$\left\| \sum_{n=1}^{\infty} n^\gamma |s_n(F) - F|^p \right\| < \infty;$$

i.e. $F \in S_p(n^\gamma)$, moreover,

$$\frac{1}{n} \sum_{\nu=1}^{n} \nu^{-(1/p)-(\gamma/p)} \leqq K\omega\left(F, \frac{1}{n}\right). \tag{3.4}$$

We show that $F \notin E(\varepsilon)$ for any ε satisfying (9). Namely, if we assume that $F \in E(\varepsilon)$ and

$$\sum_{n=1}^{\infty} n^\gamma \varepsilon_n^p < \infty \quad \text{with} \quad -1 < \gamma \leqq p-1$$

hold then (3.4) leads us to a contradiction. Indeed, these assumptions imply

$$\sum_{n=1}^{\infty} n^\gamma E_n^p(F) < \infty;$$

whence, considering the block $(n, 2n)$ in this series we infer that

$$n^{\gamma+1} E_n^p(F) \to 0. \tag{3.5}$$

Consequently, the well-known inequality

$$\omega\left(F, \frac{1}{n}\right) \leqq K \frac{1}{n} \sum_{\nu=0}^{n} E_\nu(F),$$

and (3.5) contradict (3.4) if $\gamma \leqq p-1$.

If $\gamma>p-1$, we can only give a somewhat longer proof of the statement $F\notin E(\varepsilon)$.

First we show that if $m\geqq 2$ and $2^m\leqq\nu\leqq 2^{m+1}$ then

$$|s_\nu(F,h_m)-F(h_m)|\geqq\frac{1}{4^{1+(1/p)+(\gamma/p)}}\nu^{-(1/p)-(\gamma/p)} \tag{3.6}$$

holds, where $h_m=\frac{\pi}{2^{m+4}}$.

Let $N_m=2^{m+4}$ and $\alpha=1+\frac{1}{p}+\frac{\gamma}{p}$. Then

$$F(h_m)-s_\nu(F,h_m)=\sum_{n=\nu+1}^{\infty}\frac{\sin nh_m}{n^\alpha}=\left(\sum_{n=\nu+1}^{\frac{N_m}{4}}+\sum_{n=\frac{N_m}{4}+1}^{N_m}+\sum_{n=N_m+1}^{2N_m}+\sum_{k=2}^{\infty}\sum_{n=kN_m+1}^{(k+1)N_m}\right)\frac{\sin nh_m}{n^\alpha}.$$

It is clear that for any $l\geqq 1$

$$\sum_{n=2lN_m+1}^{(2l+1)N_m}\frac{\sin nh_m}{n^\alpha}>\left|\sum_{n=(2l+1)N_m+1}^{(2l+2)N_m}\frac{\sin nh_m}{n^\alpha}\right|,$$

and thus the sum

$$\sum_{k=2}^{\infty}\sum_{n=kN_m+1}^{(k+1)N_m}\frac{\sin nh_m}{n^\alpha}$$

is positive. Furthermore we show that

$$\sum_{n=\frac{N_m}{4}+1}^{N_m}\frac{\sin nh_m}{n^\alpha}>\left|\sum_{n=N_m+1}^{2N_m}\frac{\sin nh_m}{n^\alpha}\right|.$$

It is clear that

$$\sum_{n=\frac{N_m}{2}+1}^{N_m}\frac{\sin nh_m}{n^\alpha}>\left|\sum_{n=N_m+1}^{\frac{3}{2}N_m-1}\frac{\sin nh_m}{n^\alpha}\right|,$$

and an easy calculation verifies that on account of $\alpha\geqq 1$

$$\sum_{n=\frac{N_m}{4}+1}^{N_m/2}\frac{\sin nh_m}{n^\alpha}\geqq\sum_{n=2^{m+2}+1}^{2^{m+3}}\frac{\sqrt{2}}{2n^\alpha}\geqq\frac{\sqrt{2}}{4}2^{m+3}(2^{m+3})^{-\alpha}\geqq$$

$$\geqq(2^{m+3}+1)\frac{1}{3}(2^{m+3})^{-\alpha}\geqq\left|\sum_{n=3\cdot 2^{m+3}}^{2^{m+5}}\frac{\sin nh_m}{n^\alpha}\right|=\left|\sum_{n=\frac{3}{2}N_m}^{2N_m}\frac{\sin nh_m}{n^\alpha}\right|.$$

Collecting the results we obtain that

$$F(h_m)-s_\nu(F, h_m) \geqq \sum_{n=\nu+1}^{\frac{N_m}{4}} \frac{\sin nh_m}{n^\alpha} \geqq$$

$$\geqq \sum_{n=2^{m+1}+1}^{2^{m+2}} \frac{\sin nh_m}{n^\alpha} \geqq \frac{2}{\pi} h_m \sum_{n=2^{m+1}+1}^{2^{m+2}} n^{1-\alpha} >$$

$$> 2^{-m-3} \cdot 2^{m+1} \cdot 2^{(m+2)(1-\alpha)} \geqq 4^{-\alpha} \nu^{-(1/p)-(\gamma/p)},$$

which proves (3.6).

By (3.6) we obviously obtain

$$\left\| \sum_{n=2^m+1}^{2^{m+1}} n^\gamma |s_n(F)-F|^p \right\| \geqq C > 0, \quad \text{whence} \quad \sum_{m=0}^{\infty} \left\| \sum_{n=2^m+1}^{2^{m+1}} n^\gamma |s_n - F|^p \right\| = \infty$$

and, by the mentioned equivalence theorem (see Theorem 4 of [6]),

$$\sum_{n=1}^{\infty} n^\gamma E_n^p(F) = \infty$$

follow, i.e. $F \in E(\varepsilon)$ does not hold.

Thus Theorem 2 is proved.

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